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A NEW APPROACH TO MONTE CARLO ANALYSIS ILLUSTRATED FOR RESISTIVE LADDER NETWORKS

Houssain Kettani and B. Ross Barmish
ECE Department
University of Wisconsin
Madison, WI 53706

Abstract — **In this paper, we illustrate a new approach to Monte Carlo simulation in the context of uncertain resistive ladder networks. Given lower and upper bounds on the value of each uncertain resistor but no probability distribution, we consider the problem of finding the expected value for a designated gain. In view of the fact that no apriori probability distributions for the uncertain resistors are assumed, a certain type "distributional robustness" is sought. It is seen that the expected gain which results via our new method can differ considerably from the expected gain which is obtained via a more classical approach.**

I. Introduction and Formulation

Researchers have long recognized that Monte Carlo simulation results for electrical circuits can be quite sensitive to the choices of probability distributions which are imposed on uncertain parameters; e.g., see [1], [4] and [11]. In this context, a typical question to ask is: If a circuit has an uncertain resistor whose statistics are unknown, is there justification to assume some "typical" distribution such as normal or uniform? With such questions as motivation, this paper describes a new approach to Monte Carlo simulation within the context of resistive ladder networks. In view of the fact that no apriori probability distributions are imposed on the circuit components, a certain type of "distributional robustness" is sought. For these special structures, we provide results which are stronger than those obtained for the case of an arbitrary resistive network; for example, see [10].

The conceptual framework for the new approach in this paper originates with [5]. We address the case when there is not even a partial statistical description of the uncertainty. This setup can be contrasted with other formulations; e.g., in the theory of robust statistics [6], so-called ϵ-corruptions of known density functions are assumed. Here, however, the only apriori information assumed is bounds on the uncertain parameters. In the context of this paper, such bounds are assumed for the values of the resistances R_i entering the ladder network under consideration. Whereas classical Monte Carlo theory assumes a distribution as input to the theory, the approach described herein leads to the "appropriate" distribution as the output of the theory; i.e., the theory first determines the appropriate distribution, and, only then is computer simulation carried out. Within our new framework, it becomes possible to carry out Monte Carlo simulation without knowing the probability distributions for the uncertain parameters in advance.

To be more specific, we consider a resistive ladder network consisting of an input voltage source V_{in}, an output voltage V_{out} across a designated resistor $R_{out} = R_n$ and uncertain resistors $R \doteq (R_1, R_2, \ldots, R_n)$ as depicted in Figure 1.0.1.

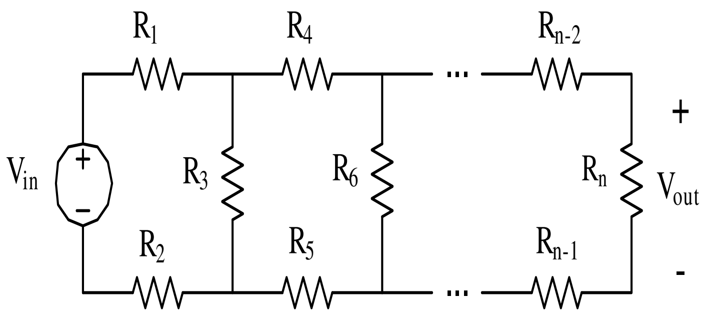

Figure 1.0.1: Ladder Network

We focus attention on the uncertain gain

$$g(R) \doteq \frac{V_{out}}{V_{in}}.$$

With each uncertain resistor R_i assumed to be an independent random variable with probability density function $f_i(R_i)$, the joint probability density function is

$$f(R) \doteq f(R_1, R_2, \ldots, R_n) = f_1(R_1) f_2(R_2) \cdots f_n(R_n)$$

and the multi-dimensional integral for the *expected gain* is given by

$$\mathcal{E}(g(R)) \doteq \int_{\mathcal{R}} f(R) g(R) dR.$$

where $\mathcal{R}$ represents the box of admissible resistor values. That is, to describe the uncertainty, for each resistor, we write

$$R_i \doteq R_{i,0} + \Delta R_i$$

with *nominal manufacturing value* $R_{i,0} > 0$ and *uncertainty* ΔR_i with given bounds

$$|\Delta R_i| \leq r_i; \;\; r_i \geq 0$$

for $i = 1, 2, \ldots, n$. Subsequently, with

$$R_i \in \mathcal{R}_i \doteq [R_{i,0} - r_i, R_{i,0} + r_i],$$

the box of *admissible resistor uncertainty* is given by

$$\mathcal{R} \doteq \mathcal{R}_1 \times \mathcal{R}_2 \times \cdots \times \mathcal{R}_n.$$

To complete the formal description of the basic circuit model, we also include the condition that only positive resistances are feasible. That is, $R_{i,0} > r_i$ for $i = 1, 2, \ldots, n$.

1.1 Probability Density Functions: Following the development in [10], it is assumed that each R_i is an independent random variable supported in $\mathcal{R}_i$ with an unknown probability density function $f_i(R_i)$ which is symmetric about its mean $R_{i,0}$ and non-increasing with respect to $|R_i - R_{i,0}|$. We write $f \in \mathcal{F}$ to denote an *admissible joint density function* $f(R)$ over $\mathcal{R}$. Given any $f \in \mathcal{F}$, the resulting random vector of resistors is denoted as R^f. Two important special cases of interest are obtained when, for some resistor R_i, $f_i = u$ is the uniform distribution or $f_i = \delta$ is the impulse (Dirac) distribution centered on $R_{i,0}$.

II. Main Result

2.1 Theorem: *Consider the multi-stage ladder network of Figure 1.0.1. For the case of maximizing $\mathcal{E}(g(R^f))$, define probability density function f^* with marginals f_i^* as follows: Set $f_i^* = \delta$ for the inter-stage resistors $(R_3, R_6, \ldots, R_n)$ and $f_i^* = u$ for the remaining resistors. Then,*

$$\mathcal{E}(g(R^{f^*})) = \max_{f \in \mathcal{F}} \mathcal{E}(g(R^f)).$$

For the case of minimizing $\mathcal{E}(g(R^f))$, define probability density function f^* with marginals f_i^* as follows: Set $f_i^* = u$ for the inter-stage resistors $(R_3, R_6, \ldots, R_n)$ and $f_i^* = \delta$ for the remaining resistors. Then,

$$\mathcal{E}(g(R^{f^*})) = \min_{f \in \mathcal{F}} \mathcal{E}(g(R^f)).$$

III. Proof of Theorem 2.1

The proof of this theorem is facilitated by the use of three preliminary lemmas.

3.1 Lemma: *The gain $g(R)$ can be expressed as*

$$g(R) = \frac{\prod_{i=1}^{\frac{n}{3}} R_{3i}}{\Delta(R_1, R_2, \ldots, R_n)}$$

where $\Delta(R_1, R_2, \ldots, R_n)$ is a multilinear function of the network resistors $R_1, R_2, \ldots, R_n$.

Proof: Using mesh analysis [8] and Cramer's rule, the output voltage can be written as

$$V_{out} = \frac{R_n \Delta'(R_1, R_2, \ldots, R_n, V_{in})}{\Delta(R_1, R_2, \ldots, R_n)}$$

where

$$\Delta'(R_1, R_2, \ldots, R_n, V_{in}) =$$

$$\begin{vmatrix} \sum_{i=1}^{3} R_i & -R_3 & 0 & \ldots & & 0 & V_{in} \\ -R_3 & \sum_{i=3}^{6} R_i & -R_6 & 0 & \ldots & & 0 \\ 0 & -R_6 & \sum_{i=6}^{9} R_i & -R_9 & 0 & \ldots & 0 \\ \vdots & \ddots & \ddots & \ddots & & & \vdots \\ 0 & \ldots & & 0 & -R_{n-6} & \sum_{i=n-6}^{n-3} R_i & 0 \\ 0 & \ldots & & & 0 & -R_{n-3} & 0 \end{vmatrix}.$$

It is now easy to obtain

$$\Delta'(R_1, R_2, \ldots, R_n, V_{in}) = V_{in} \prod_{i=1}^{\frac{(n-3)}{3}} R_{3i}.$$

This leads to the indicated numerator of the gain.

The denominator of the gain is simply the determinant of the mesh matrix. From mesh analysis, the mesh matrix has the sum of the i^{th} loop resistors in its i^{th} diagonal entry and the negative of the shared resistors between the loop i and j as its $(i,j)^{th}$ entry.

Hence, if the resistor under consideration is R_k, then the determinant of the mesh matrix Δ is of the form

$$\Delta = \begin{matrix} \\ \\ i \\ \\ j \\ \\ \end{matrix} \begin{vmatrix} & i & & j & \\ & \vdots & & \vdots & \\ \ldots & R_k + C_1 & \ldots & -R_k & \ldots \\ & \vdots & & \vdots & \\ \ldots & -R_k & \ldots & R_k + C_2 & \ldots \\ & \vdots & & \vdots & \end{vmatrix}.$$

In view of this rank one dependency on R_k, it now follows that Δ is linear in R_k. Hence the denominator of the gain is multilinear in all resistors of the circuit.

3.2 Lemma: *For the i^{th} resistor, define variables $x \doteq R_i$, $y \doteq (R_1, R_2, \ldots, R_{i-1}, R_{i+1}, \ldots, R_n)$, and denote the determinant Δ of the mesh matrix by*

$$\Delta \doteq C(y)x + D(y).$$

Then for all y associated with $R \in \mathcal{R}$, both $C(y)$ and $D(y)$ are positive.

Sketch of Proof: It first must be argued that $C(y) \neq 0$ for the ladder configuration. Subsequently, proceeding by contradiction, suppose $C(y) < 0$. Since $\Delta > 0$, this forces $D(y) > 0$. Letting

$$R_i = -\frac{D(y)}{C(y)},$$

we have

$$C(y)R_i + D(y) = 0.$$

This however, contradicts the fact that the mesh matrix is nonsingular. The proof for $D(y) > 0$ is similar.

3.3 Remarks: The lemma and its proof to follow incorporate some of the ideas introduced in [10]. In contrast to reference [10] where general resistive networks are considered, this paper provides results under the strengthened hypothesis that a ladder is being considered. In the general case, a so-called "essentiality condition" comes into play.

3.4 Lemma: *Given positive constants $c, d, t > 0$ such that $d - ct > 0$, the inequality*

$$\frac{d}{ct(d-ct)(d+ct)} > \frac{1}{2c^2t^2}\log(\frac{d+ct}{d-ct})$$

holds.

Proof: It suffices to show that

$$\frac{2cd}{t(d-ct)(d+ct)} - \frac{1}{t^2}\log(\frac{d+ct}{d-ct}) > 0. \quad (*)$$

Indeed, we define

$$z \doteq \frac{d+ct}{d-ct}$$

and note that $z \geq 1$ since $d - ct > 0$. Hence, satisfaction of $(*)$ is equivalent to

$$\frac{z}{2} - \frac{1}{2z} - \log z > 0$$

for all $z > 1$. Now letting

$$f(z) \doteq \frac{z}{2} - \frac{1}{2z} - \log z$$

and computing

$$\begin{aligned} \frac{\partial f}{\partial z} &= \frac{1}{2} + \frac{1}{2z^2} - \frac{1}{z} \\ &= \frac{(z-1)^2}{2z^2} > 0, \end{aligned}$$

it is apparent that $f(z)$ is an increasing function in z. Since $f(1) = 0$, it follows that $f(z) > 0$ for all $z > 1$. Hence, $(*)$ holds and the proof is complete.

3.5 Proof of Theorem 2.1: Holding all resistors fixed except R_i, by Lemma 3.1, the gain is of the form

$$g(R) = \begin{cases} \frac{b}{cR_i + d} & \text{i = 3k;} \\ \\ \frac{aR_i}{cR_i + d} & \text{otherwise} \end{cases}$$

where a, b, c and d are positive constants.

To handle both the inter-stage and the remaining resistors, we take

$$g(R) = \frac{aR_i + b}{cR_i + d}$$

with either $a > 0$ and $b = 0$ or $b > 0$ and $a = 0$.

In view of existing results on probabilistic robustness, for example, see [5] and [7], the next step of the proof involves noting that extremizing the expected gain can be carried out over a distinguished subset of $\mathcal{F}$ consisting of *truncated uniform distributions.* That is, for

$$t \in T \doteq [0, r_1] \times [0, r_2] \times \cdots \times [0, r_n],$$

we have

$$\max_{f \in \mathcal{F}} \mathcal{E}(g(R^f)) = \max_{t \in T} \mathcal{E}(g(R^t))$$

where R^t is the random vector with probability density function which is uniform over the *truncation box*

$$\mathcal{R}^t \doteq \mathcal{R}^{t,1} \times \mathcal{R}^{t,2} \times \cdots \times \mathcal{R}^{t,n}$$

where

$$\mathcal{R}^{t,i} \doteq [R_{i,0} - t_i, R_{i,0} + t_i].$$

Hence,

$$\max_{t \in T} \mathcal{E}(g(R^t)) = \max_{t \in T} \frac{1}{2^n t_1 t_2 \cdots t_n} \int_{\mathcal{R}^t} g(R) dR$$

with the understanding that if $t_i = 0$, the corresponding integral with $\frac{1}{2t_1}$ multiplier is calculated using an appropriate impulse distribution or L'Hopital's rule.

We now prove the part of the theorem addressing the maximum gain while noting that a similar proof can be applied for the minimum gain. Indeed, let $t^* \in T$ be the truncation corresponding to probability density functions f_i^* as prescribed in the theorem. That is, if $f_i^* = u$, then, $t_i^* = r_i$. Alternatively, if $f_i^* = \delta$, then, $t_i^* = 0$. In addition, let $t \in T$ denote any candidate truncation for the maximization of $\mathcal{E}(g(R^t))$. To show that t^* attains the maximum, it will be shown that we can replace components t_k of t with corresponding components t_k^* of t^*, one at a time, without decreasing $\mathcal{E}(g(R^t))$. For example, with $n = 3$, such a sequential replacement corresponds to the sequence of inequalities

$$\begin{aligned} \mathcal{E}(g(R^{(t_1,t_2,t_3)})) &\leq \mathcal{E}(g(R^{(t_1^*,t_2,t_3)})) \\ &\leq \mathcal{E}(g(R^{(t_1^*,t_2^*,t_3)})) \\ &\leq \mathcal{E}(g(R^{(t_1^*,t_2^*,t_3^*)})). \end{aligned}$$

That is, by showing that

$$\mathcal{E}(g(R^t)) \leq \mathcal{E}(g(R^{(t_1,t_2,\ldots,t_{k-1},t_k^*,t_{k+1},\ldots,t_n)})),$$

holds for arbitrary k, we can replace components one at a time until we arrive at the desired result

$$\mathcal{E}(g(R^t)) \leq \mathcal{E}(g(R^{t^*})).$$

Now, to separate out the dependence on R_i, we define variable
$X \doteq R_i$ and $y \doteq (R_1, R_2, \ldots, R_{i-1}, R_{i+1}, \ldots, R_n)$ and consider the conditional expectation

$$E(y, t_i) \doteq \frac{1}{2t_i} \int_{R_{i,0}-t_i}^{R_{i,0}+t_i} g(X, y) dX.$$

Claim: The inequality

$$E(y, t_i) \leq E(y, t_i^*)$$

holds for all admissible $y \in \mathcal{Y}$ where $\mathcal{Y}$ denotes the box of admissible resistor uncertainty for y.

To prove this claim, it is first noted that Lemma 3.1 and 3.2 imply that the conditional expectation under consideration is

$$\begin{aligned} E(y, t_i) &= \frac{1}{2t_i} \int_{R_{i,0}-t_i}^{R_{i,0}+t_i} \frac{A(y)X + B(y)}{C(y)X + D(y)} dX \\ &= \frac{1}{2t_i} \int_{-t_i}^{t_i} \frac{ax + b}{cx + d} dx \\ &= \frac{a}{c} + \frac{bc - da}{2t_i c^2} \log \frac{d + ct_i}{d - ct_i} \end{aligned}$$

where $a(y) \doteq A(y), b(y) \doteq A(y)R_{i,0} + B(y), c(y) \doteq C(y), d(y) \doteq C(y)R_{i,0} + D(y)$. Furthermore, we obtain partial derivative computed to be

$$\frac{\partial E}{\partial t_i} = (bc - ad)e(t_i)$$

where

$$e(t_i) \doteq \frac{d}{ct_i(d - ct_i)(d + ct_i)} - \frac{1}{2c^2 t_i^2} \log(\frac{d + ct_i}{d - ct_i}).$$

In view of Lemma 3.4, the inequality $e(t_i) > 0$ holds.
Hence, for the inter-stage resistors, $b = 0$ leads to $bc - ad = -ad < 0$, and

$$\frac{\partial E}{\partial t_i} < 0.$$

Therefore, $E(t_i, y)$ is maximized at $t_i = t_i^* = 0$. If R_i is not an inter-stage resistor, however, then $a = 0$ leads to $bc - ad = bc > 0$, and

$$\frac{\partial E}{\partial t_i} > 0.$$

Hence, $E(t_i, y)$ is maximized at $t_i = t_i^* = r_i$. This completes the proof of the claim.

Finally, to complete the proof of the theorem, we now observe that it follows from the claim that

$$\begin{aligned} \mathcal{E}(g(R^t)) &= \frac{1}{2^{n-1} t_1 t_2 \cdots t_{i-1} t_{i+1} \cdots t_n} \int_{\mathcal{Y}} E(y, t_i) dy \\ &\leq \frac{1}{2^{n-1} t_1 t_2 \cdots t_{i-1} t_{i+1} \cdots t_n} \int_{\mathcal{Y}} E(y, t_i^*) dy \\ &= \mathcal{E}(g(R^{(t_1, t_2, \ldots, t_{i-1}, t_i^*, t_{i+1}, \ldots, t_n)})). \end{aligned}$$

IV. Example

The ideas above are now illustrated for a three stage network considered in [10], with nominal values $R_{1,0} = R_{4,0} = R_{5,0} = R_{7,0} = R_{8,0} = 1$, $R_{2,0} = 2$, $R_{3,0} = 3$, $R_{6,0} = 5$ and $R_{9,0} = 7$. To provide a case where a classical Monte Carlo simulation yields a maximum expected gain which differs dramatically from the one obtained via the methods in this paper, we considered uncertainty bounds equal to 80% of the nominal for the inter-stage resistors and 10% for the remaining resistors. As prescribed by Theorem 2.1, we carried out a Monte Carlo simulation using an impulsive distribution for R_3, R_6 and R_9 and a uniform distribution for the remaining R_i. With 100,000 samples, we obtained the estimate

$$\mathcal{E}(g(R^{f^*})) \approx 0.1864.$$

Next, a classical Monte Carlo simulation using the uniform distribution for all resistors was carried out. This time, an estimate

$$\mathcal{E}(g(R^u)) \approx 0.1554.$$

was obtained. In conclusion, it is apparent that the classical expected gain is less than the distributionally robust expected gain by about 20%. We also note that in both cases the expectation converges; see Figure 4.0.1 for the convergence plots corresponding to the optimal and uniform distributions respectively.

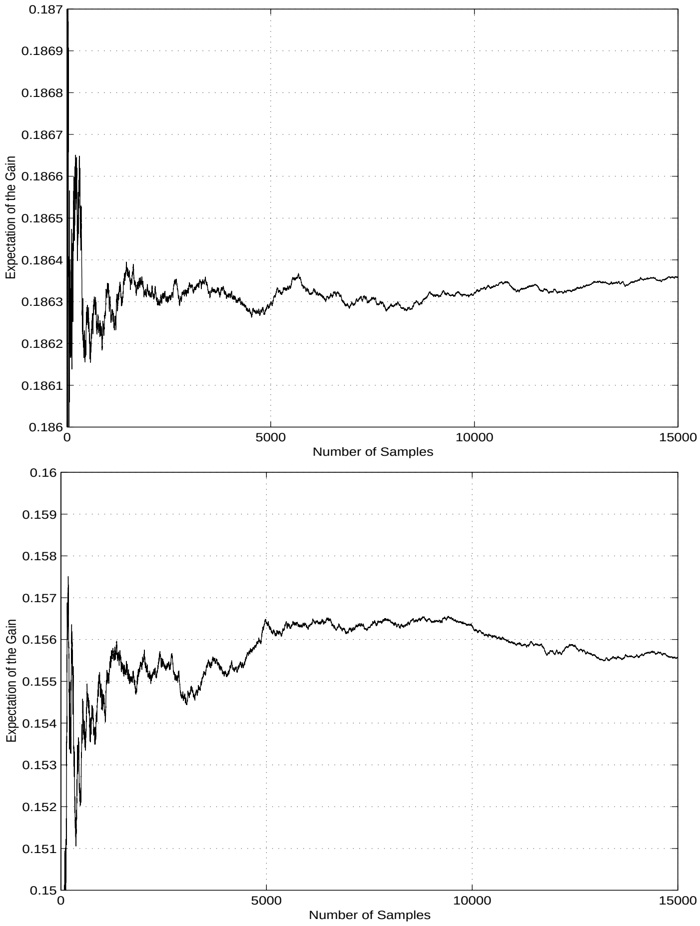

Figure 4.0.1: Convergence of Expectation

V. Concluding Remarks

For the example above, it is noted that the maximum gain can be obtained by setting $R_1 = R_4 = R_5 = R_7 = R_8 = 0.9$, $R_2 = 1.8$, $R_3 = 5.4$, $R_6 = 9$ and $R_9 = 12.6$, and

$$\max_{R \in \mathcal{R}} g(R) \approx 0.3535.$$

While the minimum gain can be obtained by setting $R_1 = R_4 = R_5 = R_7 = R_8 = 1.1$, $R_2 = 2.2$, $R_3 = 0.6$, $R_6 = 1$ and $R_9 = 1.4$, and

$$\min_{R \in \mathcal{R}} g(R) \approx 0.0134,$$

this range for the worst-case gain values is seen to differ significantly from the expected gain values; e.g., notice that

$$\frac{\max_{R \in \mathcal{R}} g(R)}{\max_{f \in \mathcal{F}} \mathcal{E}(g(R^f))} \approx 1.8965$$

More generally, the lemma below provides the basis for such comparisons.

5.1 Lemma: *Consider the multi-stage ladder network of Figure 1.0.1. For the case of maximizing $g(R)$, define R^* with components $R_i^* = R_{i,0} + r_i$ for the inter-stage resistors $(R_3, R_6, \ldots, R_n)$ and $R_i^* = R_{i,0} - r_i$ for the remaining resistors. Then,*

$$g(R^*) = \max_{R \in \mathcal{R}} g(R).$$

For the case of minimizing $g(R)$, define R^* with components $R_i^* = R_{i,0} - r_i$ for the inter-stage resistors $(R_3, R_6, \ldots, R_n)$ and $R_i^* = R_{i,0} + r_i$ for the remaining resistors. Then,

$$g(R^*) = \min_{R \in \mathcal{R}} g(R).$$

Proof: Note that the gain is non-decreasing with respect to the inter-stage resistors, and non-increasing with respect to the remaining resistors, since $a, b, c, d > 0$. Hence the result is easily obtained.

5.2 Remarks: The comparison of the worst-case and expected gain values above suggests one direction for further research. That is, for larger and more general resistive networks, new theory facilitating such comparisons would be useful in the evaluation of risk associated with various degrees of uncertainty.

A second area for further research involves generalization from the ladder network to rather arbitrary networks. A fundamental problem is to characterize $f^* \in \mathcal{F}$ maximizing or minimizing the expected value of some designated gain. To this end, it is often quit possible to obtain results along the lines given here for non-ladder configurations.

To illustrate how the ideas in this paper apply to networks in a non-ladder context, we consider the circuit in Figure 5.2.1. With gain $g(R)$ obtained using the voltage across R_8 as output, we now solve the problem of assigning probability density functions to each resistor. Indeed, for gain maximization, using analysis similar to that used in the proof of Theorem 2.1, the following result is obtained: For resistors R_3 and R_8, the Dirac delta function is assigned and for the remaining resistors, a uniform distribution is assigned. With this distinguished assignment of probability density functions $f^* \in \mathcal{F}$, it can be shown that any other assignment leads to a smaller value of the expected gain.

Finally, is interesting to note that this assignment of density functions leads to the maximum expected gain no matter what nominal values $R_{i,0}$ are assumed by the resistors and no matter what uncertainty bound $r_i \leq R_{i,0}$ is used. In terms of the theory developed in [10], all resistors in this network satisfy the so-called essentiality condition.

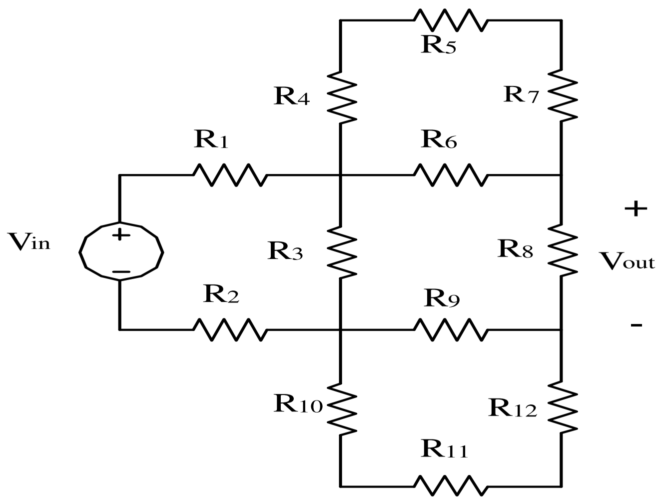

Figure 5.2.1: Non-Ladder Network

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