

# A Functional Conversion from Decimal to Base $b$ Numbers

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## Abstract

*We consider the problem of converting a decimal number to a base  $b$  number. We present a conversion function that relates each digit in the base  $b$  system to the decimal value that is equal to the base  $b$  number in question. Thus, each base  $b$  digit of the related base  $b$  number can be obtained directly from the corresponding decimal number without the requirement of knowing any other base  $b$  digit.*

## 1 Introduction

We represent an unsigned  $b$ -ary, radix  $b$ , or base  $b$  number as the following string of digits:

$$(d_n d_{n-1} \dots d_0 . d_{-1} d_{-2} \dots d_{-m+1} d_{-m})_b,$$

where the integer  $b \geq 2$  and  $d_j \in \{0, 1, \dots, b-1\}$  for  $j = n, n-1, \dots, -m+1, -m$ , and “.” is the radix point. The term to the left of the radix point is referred to as the integer part, while that to the right of the radix point is referred to as the fractional part. The everyday number system that we use is the decimal (base 10). Other number systems that are used in computer work are binary (base 2), octal (base 8), and hexadecimal (base 16).

The classical way to convert a base  $b_1$  number to another number in base  $b_2$ , so that both have the same decimal value, is to first convert base  $b_1$  to decimal then convert from the latter to base  $b_2$  — for example, see [1].

A more convenient conversion method is adopted when  $b_1 = b_2^p$  where  $p$  is a positive decimal integer. Then, to convert from base  $b_2$  to base  $b_1$ , we gather  $p$  base  $b_1$  digits (from right to left for the integer part, and from left to right for the fraction part) into one base  $b_2$  digit so that both have the same decimal value. On the other hand, to convert from base  $b_1$  to base  $b_2$ , we expand one base  $b_2$  digit into  $p$  base  $b_1$  digits so that both have the same decimal value. In other words, the relation between base  $b$  and base  $b^p$  is given by

$$(\dots d_1 d_0 . d_{-1} \dots)_b = (\dots d'_1 d'_0 . d'_{-1} \dots)_{b^p},$$

where

$$d'_j = (d_{pj+p-1} \dots d_{pj+1} d_{pj})_b.$$

To illustrate this point, consider conversion between binary and hexadecimal. In this case we have  $b_2 = 2$  and  $p = 4$ . As an illustrative numerical example, we have  $(00100101)_2 = (25)_{16}$ .

Suppose now we would like to write

$$(d)_{10} = (d_n d_{n-1} \dots d_0 . d_{-1} d_{-2} \dots d_{-m+1} d_{-m})_b,$$

where the right hand side is a decimal number and the left hand side is a base  $b$  number. Then, by definition, to convert from base  $b$  to decimal, we write

$$d = \sum_{j=-m}^n d_j b^j.$$

Alternatively, the classical way to convert from decimal to base  $b$ , is to repeatedly divide the integer part of  $d$  by  $b$  and record the remainders. Then, starting with the first recorded one, we write down these remainders from right to left starting to the left of the radix point. Thus, we obtain the integer part of the corresponding base  $b$  number.

On the other hand, the fraction part is obtained as follows. We repeatedly multiply the fraction part of  $d$  by  $b$  and record the resultant integer. Then, starting with the first recorded one, we write down these integers from left to right starting to the right of the radix point. Thus, we obtain the fraction part of the corresponding base  $b$  number — see [1] for details and illustrative examples.

However, is there any simple function that we can use to convert from decimal to base  $b$ ? Moreover, what if we are only interested in the value of  $d_j$  for some  $j \geq 0$ ? Then based on the repeated division algorithm, we need to compute all the values  $d_k$  for  $k = 0, 1, \dots, j$ . Also, what if we are only interested in the value of  $d_j$  for some  $j < 0$ ? Then based on the repeated multiplication algorithm, we need to compute all the values  $d_k$  for  $k = -1, -2, \dots, j$ .

We answer the above questions and present a very simple and useful result in the following section.

## 2 Conversion between Decimal and Base $b$ Numbers

The following theorem answers the questions posed in the introduction and presents a very simple and useful result. By this theorem, the  $d_j$ 's are directly accessible without the requirement of computing any other  $d_j$  except the one of interest. In this theorem, we use  $\lfloor \cdot \rfloor$  to indicate the floor function.

### 2.1 Theorem

*Let  $d \geq 0$  be a decimal number and  $b \geq 2$  be a decimal integer. Let also  $d_j \in \{0, 1, \dots, b-1\}$  for  $j = n, n-1, \dots, -m+1, -m$ . If  $(d)_{10} = (d_n d_{n-1} \dots d_0 . d_{-1} d_{-2} \dots d_{-m+1} d_{-m})_b$ , then  $d_j = \lfloor db^{-j} \rfloor - b \lfloor db^{-j-1} \rfloor$*

### 2.2 Proof of Theorem

Before proving the theorem, we adopt the following generalization of the divisibility concept.

**Generalized Divisibility:** *Let  $a_1$  and  $a_2$  be two real numbers. We say  $a_1$  is divisible by  $a_2$  if the ratio  $a_1/a_2$  is an integer.*

Now, to prove the theorem we first let  $y = db^{-j-1}$ , and consider two cases:

**Case 1:**  $y \in \mathbb{N}$ . This means that  $d$  is divisible by  $b^{j+1}$ . This in turn, forces  $d_k = 0$  for  $k \leq j$ . Accordingly,  $\lfloor by \rfloor - b \lfloor y \rfloor = 0$ .

**Case 2:**  $y \notin \mathbb{N}$ . Then we can write  $y = x + \epsilon$ , where  $x \in \mathbb{N}$  and  $\epsilon \in (0, 1)$ . Thus,

$$\lfloor by \rfloor - b \lfloor y \rfloor = \lfloor b\epsilon \rfloor.$$

On the other hand, we can write

$$d = xb^{j+1} + \epsilon b^{j+1}.$$

Now, the term  $xb^{j+1}$  would modify  $d_k$  for some  $k > j$ . To see the effect on  $d_j$ , we need to

observe the term  $\epsilon b^{j+1}$ . To do this, we partition the interval  $(0, 1)$  into  $b$  subintervals of equal length. Thus, we consider the following cases  $\epsilon \in [\frac{b-i}{b}, \frac{b-i+1}{b})$ ,  $i = b, b-1, \dots, 1$ . Therefore, we have  $(b-i)b^j \leq \epsilon b^{j+1} < (b-i+1)b^j$ , which implies that  $d_j = b-i$ . Accordingly,  $\lfloor b\epsilon \rfloor = b-i$ . This concludes the proof.

Note that it can easily be shown that the values of the  $d_j$ 's are also equal to  $\lfloor b(db^{-j-1} - \lfloor db^{-j-1} \rfloor) \rfloor$  and  $\lfloor b^{-j} \bmod(d, b^{j+1}) \rfloor$ .

### 3 Summary

We have discussed conversion between any two number systems. Unlike the classical conversion from decimal to base  $b$  system, we have presented a functional conversion that expresses each digit in base  $b$  in terms of the decimal number in question. This in turn, provides us with a direct access to each base  $b$  digit instead of the need of knowing any previous base  $b$  digits. Finally, we note that the result of this paper can prove to be very useful tool in the fields of error detection, parallel computing, cryptography, and steganography.

### References

- [1] M. M. Mano and C. R. Kime (2001), "*Logic and Computer Design Fundamentals*," second edition, Prentice Hall.