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Contributions to the Theory of the Non-Central χ^2 Distribution

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Abstract

In this paper, we consider the probability density function (pdf) of a non-central χ^2 distribution with odd number of degrees of freedom ν. This pdf is represented in the literature as an infinite sum. Consequently, we present three alternative expressions to this pdf. The first expression is in terms of a partial derivative of the hyperbolic cosine function. The second expression, on the other hand, is a finite sum representation of $(\nu+1)/2$ terms only instead of the infinite sum. Finally, we present a general recurrence relation for such pdf. These results have applications in approximation of the pdf of non-central χ^2 distributed random variables.

1 Introduction

A non-central χ^2 distribution with ν degrees of freedom is the distribution of the sum of the squares of ν random variables that are normally distributed with unit variance and nonzero means. In other words, let $x_i \sim N(\mu_i, 1)$, and $y = \sum_{i=1}^{\nu} x_i^2$. Then the distribution of y is a non-central χ^2 with ν degrees of freedom and non-centrality parameter $\lambda = \sum_{i=1}^{\nu} \mu_i^2$. The probability density function (pdf) of such distribution is expressed as

$$f_\nu^\lambda(y) = 2^{-\frac{\nu}{2}} \exp[-\frac{1}{2}(y+\lambda)] \sum_{i=0}^{\infty} \frac{y^{(\nu/2)+i-1}\lambda^i}{\Gamma(\frac{\nu}{2}+i)2^{2i}i!}, \tag{1}$$

where $\Gamma(\cdot)$ is the Gamma function.

When $\lambda = 0$, (equivalently, the means μ_i are zero), then this distribution is reduced to the central χ^2 or simply χ^2. The pdf of this distribution is given by

$$f_\nu^0(y) = \frac{y^{(\nu/2)-1}\exp(-y/2)}{2^{\nu/2}\Gamma(\nu/2)}.$$

Thus, the pdf of the non-central χ^2 distribution can be expressed as an infinite weighted sum of central χ^2 pdf as follows:

$$f_\nu^\lambda(y) = \sum_{i=0}^{\infty} p^{\lambda/2}(i) f_{\nu+2i}^0(y),$$

where $p^\theta(i) = e^{-\theta}\theta^i/i!$ is the Poisson pdf with parameter θ (see [3]).

Another expression for the pdf of the non-central χ^2 that does not explicitly involve an infinite sum is given by

$$f_\nu^\lambda(y) = \exp[-\frac{1}{2}(\lambda + y)]\frac{1}{2}\left(\frac{y}{\lambda}\right)^{(\nu-2)/4} I_{(\nu-2)/2}(\sqrt{\lambda y}) \tag{2}$$

where $I_\alpha(\cdot)$ is the modified Bessel function of the first kind of degree α and nonetheless is given by the following infinite sum

$$I_\alpha(y) = (y/2)^\alpha \sum_{i=0}^{\infty} \frac{(y/2)^{2i}}{i!\Gamma(\alpha + i + 1)}. \tag{3}$$

We note here though, that the function $I_\alpha(y)$ is built in scientific computing programs such as MATLAB. We refer the reader to [2, pp. 900 – 932] for more information on various kinds of Bessel functions and some of the identities and approximations associated with them. The reader is also referred to Chapter 29 of [3] for detailed discussion on non-central χ^2 distribution.

Because of its close association with the normal distribution, non-central χ^2 distribution arises frequently in various applications including finance, estimation theory, decision theory, and time series analysis (see [4] for examples and details). In addition, and for numerical evaluation purpose, the infinite sum in the pdf of non-central χ^2 distribution tends to be approximated by a finite sum.

In this paper, we consider the case when the degree of freedom ν is odd, and present three different representations of the pdf. The first is in terms of the partial derivative of the hyperbolic cosine function, the second is a finite sum representation instead of the infinite sum in (1) and the third is a general recurrence. These results are presented in Section 2. Concluding remarks and further research direction is presented in Section 3. An index of notations is presented at the end of this paper to facilitate understanding and look up of notations used in this paper.

2 Alternative Expressions

In this section, we present and prove several new theorems that set forth alternative expressions to the pdf of a non-central χ^2 distribution when the number of degrees of freedom is odd. Accordingly, Theorem 2.1 expresses the pdf in terms of the n^{th} partial derivative of the hyperbolic cosine function. Theorem 2.2 presents a finite sum representation of the modified Bessel function instead of the infinite sum in (3). This result is needed to prove Theorem 2.3 which presents a finite sum representation of the pdf that consists of $(\nu + 1)/2$ terms only instead of the infinite sum in (1). This section ends with Theorems 2.4 and 2.5 that present general recurrence relations for the pdf.

2.1 Theorem

For $\nu = 2n + 1$, $n \in \mathbb{N}$, the pdf of a non-central χ^2 distribution is given by

$$f(y) = (2y/\lambda)^n \frac{\exp[-\frac{1}{2}(y + \lambda)]}{\sqrt{2\pi y}} \frac{\partial^n \cosh(\sqrt{\lambda y})}{\partial y^n}. \tag{4}$$

Proof: We first write

$$f(y) = \exp[-\frac{1}{2}(y+\lambda)]g(y),$$

where,

$$g(y) = 2^{-\frac{\nu}{2}} \sum_{j=0}^{\infty} \frac{y^{(\nu/2)+j-1}\lambda^j}{\Gamma(\frac{\nu}{2}+j)2^{2j}j!}.$$

We next substitute $\nu = 2n+1$ and get

$$g(y) = 2^{-n-\frac{1}{2}} \sum_{j=0}^{\infty} \frac{y^{n+j-\frac{1}{2}}\lambda^j}{\Gamma(n+j+\frac{1}{2})2^{2j}j!}.$$

Next we use the identity (see [2, p.888])

$$\Gamma(n+\frac{1}{2}) = \frac{(2n)!\sqrt{\pi}}{2^{2n}n!},$$

and get

$$g(y) = (1/\sqrt{\pi})(2y)^{n-\frac{1}{2}} \sum_{j=0}^{\infty} \frac{(y\lambda)^j(n+j)!}{[2(n+j)]!j!}.$$

We then make the following change of variables $k = n+j$, and get

$$g(y) = (1/\sqrt{\pi})(2y)^{n-\frac{1}{2}} \sum_{k=n}^{\infty} \frac{(y\lambda)^{k-n}k!}{(2k)!(k-n)!}.$$

Note now that the Maclaurin series expansion of the hyperbolic cosine function is given by (see [2, p.41])

$$\cosh(y) = \sum_{j=0}^{\infty} \frac{y^{2j}}{(2j)!}.$$

Thus, it can be shown by induction that

$$\frac{\partial^n \cosh(\sqrt{\lambda y})}{\partial y^n} = \sum_{k=n}^{\infty} \frac{k!\lambda^k y^{k-n}}{(2k)!(k-n)!}.$$

Therefore,

$$g(y) = \frac{(2y/\lambda)^n}{\sqrt{2\pi y}} \frac{\partial^n \cosh(\sqrt{\lambda y})}{\partial y^n},$$

and this concludes the proof.

In the next theorem, we present a new expression for the modified Bessel function of the first kind when the degree $\alpha = (-1)^j(n+\frac{1}{2})$.

2.2 Theorem

Let $\alpha = (-1)^j(n+\frac{1}{2})$, where n is a non-negative integer. Then the modified Bessel function of the first kind is given by

$$I_{(-1)^j(n+\frac{1}{2})}(z) = \sqrt{\frac{2}{\pi z}} \sum_{i=0}^{n} (-1)^i (2i-1)!! C_{n-i}^{n+i} z^{-i} \cosh_{n-i+j+1}(z). \tag{5}$$

Proof: A finite sum representation of the modified Bessel function of the first kind is presented in [2] as follows:

$$I_{\pm(n+\frac{1}{2})}(z) = \frac{1}{\sqrt{2\pi z}} \left(e^z \sum_{i=0}^{n} \frac{(-1)^i (n+i)!}{i!(n-i)!(2z)^i} \pm (-1)^{n+1} e^{-z} \sum_{i=0}^{n} \frac{(n+i)!}{i!(n-i)!(2z)^i} \right).$$

Now by substituting $(-1)^j$ for $\pm$ and rearranging the terms, we get

$$\begin{aligned} I_{(-1)^j(n+\frac{1}{2})}(z) &= \frac{1}{\sqrt{2\pi z}} \sum_{i=0}^{n} \frac{(-1)^i (n+i)!}{i!(n-i)!(2z)^i} (e^z + (-1)^{n-i+j+1} e^{-z}) \\ &= \sqrt{\frac{2}{\pi z}} \sum_{i=0}^{n} (-1)^i (2i-1)!! C_{n-i}^{n+i} z^{-i} \cosh_{n-i+j+1} z. \end{aligned}$$

The next theorem presents a finite sum expression of the non-central χ^2 distribution when the number of degrees of freedom is odd.

2.3 Theorem

For $\nu = 2n+1$, $n \in \mathbb{N}$, the non-central χ^2 distribution is given by

$$f_{2n+1}^{\lambda}(y) = \begin{cases} \frac{\exp[-\frac{1}{2}(y+\lambda)]}{\sqrt{2\pi y}} \cosh(\sqrt{\lambda y}) & \text{if} \quad n = 0; \\ \frac{\exp[-\frac{1}{2}(y+\lambda)]}{\sqrt{2\pi y}} \left(\frac{y}{\lambda}\right)^{\frac{n}{2}} \sum_{i=0}^{n-1} (-1)^i (\lambda y)^{-\frac{i}{2}} (2i-1)!! C_{n-1-i}^{n-1+i} \cosh_{n-i} \sqrt{\lambda y} & \text{if} \quad n \geq 1. \end{cases}$$

Proof: The case $n = 0$ follows from Theorem 2.1. When $n \geq 1$, we substitute $\nu = 2n+1$ in (2) and get

$$f_{2n+1}^{\lambda}(y) = \exp[-\frac{1}{2}(\lambda + y)] \frac{1}{2} \left(\frac{y}{\lambda}\right)^{(2n-1)/4} I_{n-\frac{1}{2}}(\sqrt{\lambda y}).$$

Thus, the result follows from Theorem 2.2, and this concludes the proof.

The next two theorems present a general recurrence relation for the pdf of the non-central χ^2 distribution with odd number of degrees of freedom. Writing $\nu = 2n+1$, The first theorem presents such recurrence in the case when n is even ($n = 2p$), while the second theorem presents such recurrence in the case when n is odd ($n = 2p+1$). We note here that [1] pointed out the existence of such recurrence but failed to provide its general expression.

2.4 Theorem

For $\nu = 4p+1$, p is an integer with $p \geq 1$, the following recurrence holds true for the non-central χ^2 distribution

$$f_{4p+1}^{\lambda}(y) = g_{1,2p}^{\lambda}(y) f_1^{\lambda}(y) - g_{3,2p}^{\lambda}(y) f_3^{\lambda}(y)$$

where

$$g_{1,2p}^{\lambda}(y) = \sum_{i=0}^{p-1} \lambda^{-p-i} y^{p-i} (4i-1)!! C_{2(p-i)-1}^{2(p+i)-1}$$

and

$$g_{3,2p}^{\lambda}(y) = \sum_{i=0}^{p-1} p \lambda^{-p-i} y^{p-i-1} (4i+1)!! C_{2(p-i-1)}^{2(p+i)}.$$

Proof: First note from Theorem 2.3 that

$$f_1^{\lambda}(y) = \frac{\exp[-\frac{1}{2}(y+\lambda)]}{\sqrt{2\pi y}} \cosh(\sqrt{\lambda y}),$$

and

$$f_3^{\lambda}(y) = \frac{\exp[-\frac{1}{2}(y+\lambda)]}{\sqrt{2\pi \lambda}} \sinh(\sqrt{\lambda y}).$$

Next, for $n = 2p$ with $p \geq 1$, we have from Theorem 2.3

$$f_{4p+1}^{\lambda}(y) = \frac{\exp[-\frac{1}{2}(y+\lambda)]}{\sqrt{2\pi y}} \left(\frac{y}{\lambda}\right)^p h(y),$$

where

$$\begin{aligned} h(y) &= \sum_{i=0}^{2p-1} (-1)^i (\lambda y)^{-\frac{i}{2}} (2i-1)!! C_{2p-1-i}^{2p-1+i} \cosh_{2p-i} \sqrt{\lambda y} \\ &= \sum_{\substack{i=0 \\ i \text{ even}}}^{2p-1} (\lambda y)^{-\frac{i}{2}} (2i-1)!! C_{2p-1-i}^{2p-1+i} \cosh_{2p-i} \sqrt{\lambda y} - \sum_{\substack{i=0 \\ i \text{ odd}}}^{2p-1} (\lambda y)^{-\frac{i}{2}} (2i-1)!! C_{2p-1-i}^{2p-1+i} \cosh_{2p-i} \sqrt{\lambda y}. \end{aligned}$$

Now, substitute $i = 2k$, and $i = 2k+1$ when i is even and odd, respectively, and get

$$h(y) = \cosh \sqrt{\lambda y} \sum_{k=0}^{p-1} (\lambda y)^{-k} (4k-1)!! C_{2(p-k)-1}^{2(p+k)-1} - \sinh \sqrt{\lambda y} \sum_{k=0}^{p-1} (\lambda y)^{-k-\frac{1}{2}} (4k+1)!! C_{2(p-k-1)}^{2(p+k)}.$$

Thus,

$$f_{4p+1}^{\lambda}(y) = g_{1,2p}^{\lambda}(y) f_1^{\lambda}(y) - g_{3,2p}^{\lambda}(y) f_3^{\lambda}(y),$$

where, $g_{1,2p}^{\lambda}(y)$ and $g_{3,2p}^{\lambda}(y)$ as in the theorem.

2.5 Theorem

For $\nu = 4p+3$, p is an integer with $p \geq 1$, the following recurrence holds true for the non-central χ^2 distribution

$$f^{\lambda}_{4p+3}(y) = g^{\lambda}_{3,2p+1}(y) f^{\lambda}_{3}(y) - g^{\lambda}_{1,2p+1}(y) f^{\lambda}_{1}(y),$$

where

$$g^{\lambda}_{1,2p+1}(y) = \sum_{i=0}^{p-1} \lambda^{-p-i-1} y^{p-i} (4i-1)!! C^{2(p+i)+1}_{2(p-i)-1}$$

and

$$g^{\lambda}_{3,2p+1}(y) = \sum_{i=0}^{p} \lambda^{-p-i} y^{p-i} (4i+1)!! C^{2(p+i)}_{2(p-i)}.$$

Proof: First note from Theorem 2.3 that

$$f^{\lambda}_{1}(y) = \frac{\exp[-\frac{1}{2}(y+\lambda)]}{\sqrt{2\pi y}} \cosh(\sqrt{\lambda y}),$$

and

$$f^{\lambda}_{3}(y) = \frac{\exp[-\frac{1}{2}(y+\lambda)]}{\sqrt{2\pi \lambda}} \sinh(\sqrt{\lambda y}).$$

Next, for $n = 2p+1$ with $p \geq 1$, we have from Theorem 2.3

$$f^{\lambda}_{4p+3}(y) = \frac{\exp[-\frac{1}{2}(y+\lambda)]}{\sqrt{2\pi y}} \left(\frac{y}{\lambda}\right)^{p+\frac{1}{2}} h(y),$$

where

$$\begin{aligned} h(y) &= \sum_{i=0}^{2p} (-1)^i (\lambda y)^{-\frac{i}{2}} (2i-1)!! C^{2p+i}_{2p-i} \cosh_{2p+1-i} \sqrt{\lambda y} \\ &= \sum_{\substack{i=0 \\ i\ even}}^{2p} (\lambda y)^{-\frac{i}{2}} (2i-1)!! C^{2p+i}_{2p-i} \cosh_{2p+1-i} \sqrt{\lambda y} - \sum_{\substack{i=0 \\ i\ odd}}^{2p} (\lambda y)^{-\frac{i}{2}} (2i-1)!! C^{2p+i}_{2p-i} \cosh_{2p+1-i} \sqrt{\lambda y}. \end{aligned}$$

Now, substitute $i = 2k$, and $i = 2k+1$ when i is even and odd, respectively, and get

$$h(y) = \sinh \sqrt{\lambda y} \sum_{k=0}^{p} (\lambda y)^{-k} (4k-1)!! C^{2(p+k)}_{2(p-k)} - \cosh \sqrt{\lambda y} \sum_{k=0}^{p-1} (\lambda y)^{-k-\frac{1}{2}} (4k+1)!! C^{2(p+k)+1}_{2(p-k)-1}.$$

Thus,

$$f^{\lambda}_{4p+3}(y) = g^{\lambda}_{3,2p+1}(y) f^{\lambda}_{3}(y) - g^{\lambda}_{1,2p+1}(y) f^{\lambda}_{1}(y),$$

where, $g^{\lambda}_{1,2p+1}(y)$ and $g^{\lambda}_{3,2p+1}(y)$ as in the theorem.

3 Conclusion

We presented three finite expressions for the probability density distribution (pdf) of a non-central χ^2 distribution in the case when the number of degrees of freedom ν is odd: a finite partial derivative, a finite sum, and general recurrence expressions. The result of this paper helps finding the exact value of the distribution instead of appealing to various approximation algorithms known in the literature to approximate the infinite sum representation. It would be of interest to extend this result to the case when ν is even. So far no such expressions are known when the number of degrees of freedom is even.

Index of Notations

- $n!!$ is the double factorial function defined as

$$n!! = \begin{cases} n(n-2)\ldots 3\cdot 1 & \text{if} \quad n > 0 \quad \text{is odd;} \\ n(n-2)\ldots 4\cdot 2 & \text{if} \quad n > 0 \quad \text{is even,} \\ 1 & \text{if} \quad n = -1, 0. \end{cases}$$

- C_i^j where i and j non-negative integers with $j \geq i$ denotes the binomial coefficient defined as

$$C_i^j = \frac{j!}{i!(j-i)!}.$$

- $\cosh_i(z)$ is the alternate hyperbolic cosine/sine function defined as

$$\begin{aligned} \cosh_i(z) &= \begin{cases} \cosh(z) & \text{if} \quad i \quad \text{is even;} \\ \sinh(z) & \text{if} \quad i \quad \text{is odd,} \end{cases} \\ &= \frac{e^z + (-1)^i e^{-z}}{2}. \end{aligned}$$

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