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On the Non-Central χ^2 Distribution With Odd Number of Degrees of Freedom

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Abstract

In the literature, the probability density function (pdf) of a non-central χ^2 distribution is represented using an infinite sum. In this paper, the case when the number of degrees of freedom ν is odd is considered. Consequently, the corresponding pdf is represented as a finite sum of $(\nu+1)/2$ terms only.

I. Introduction

A non-central χ^2 distribution with ν degrees of freedom is the distribution of the sum of the squares of ν random variables that are normally distributed with unit variance and nonzero mean. In other words, let $x_i \sim N(\mu_i, 1)$, and $y = \sum_{i=1}^{\nu} x_i^2$. Then the distribution of y is a non-central χ^2 with ν degrees of freedom and non-centrality parameter $\lambda = \sum_{i=1}^{\nu} \mu_i^2$. The probability density function (pdf) of such distribution is expressed as

$$f(y) = 2^{-\frac{\nu}{2}} \exp[-\frac{1}{2}(y+\lambda)] \sum_{j=0}^{\infty} \frac{y^{(\nu/2)+j-1}\lambda^j}{\Gamma(\frac{\nu}{2}+j)2^{2j}j!}. \tag{1}$$

When $\lambda = 0$, (equivalently, the means μ_i are zero), then this distribution is reduced to the central χ^2 or simply χ^2. The pdf of this distribution is given by

$$f_c(y) = \frac{y^{(\nu/2)-1}\exp(-y/2)}{2^{\nu/2}\Gamma(\nu/2)},$$

where $\Gamma(\cdot)$ is the Gamma function.

The pdf in (1) can also be expressed as

$$f(y) = \exp(-\frac{1}{2}(\lambda+y))\frac{1}{2}\left(\frac{y}{\lambda}\right)^{(\nu-2)/4} I_{(\nu-2)/2}(\sqrt{\lambda y}) \tag{2}$$

where $I_\alpha(\cdot)$ is the modified Bessel function of the first kind of degree α and is given by the following infinite sum

$$I_\alpha(y) = (y/2)^\alpha \sum_{i=0}^{\infty} \frac{(y/2)^{2i}}{i!\Gamma(\alpha+i+1)}. \tag{3}$$

See [1, pp. 900 – 932] for more information on various kinds of Bessel functions and some of the identities and approximations associated with them. See also Chapter 29 of [2] for detailed discussion on noncentral χ^2 distribution.

In this paper, we consider the case when the degree of freedom ν is odd, and reduce the infinite sum in (1) to a finite sum.

II. Main Result

The layout of this section is as follows. Theorem II-A expresses the pdf in terms of the n^{th} partial derivative of the hyperbolic cosine function. Theorem II-B on the other hand, expresses the same pdf as a finite sum of alternating hyperbolic sine and cosine functions together with a two-dimensional recurrence relation. This recurrence is solved in Lemma II-C. These two theorems and the lemma are necessary to facilitate the proof of the main result. The section is ended by the main result in Theorem II-D, which expresses the pdf as the finite sum of elementary functions.

A. Partial Derivative Theorem

For $\nu = 2n+1$, $n \in \mathbb{N}$, the pdf of a non-central χ^2 distribution is given by

$$f(y) = \frac{\exp[-\frac{1}{2}(y+\lambda)]}{\sqrt{2\pi y}}(2y/\lambda)^n \frac{\partial^n \cosh(\sqrt{\lambda y})}{\partial y^n}. \tag{4}$$

1) Proof: We first write

$$f(y) = \exp[-\frac{1}{2}(y+\lambda)]g(y),$$

where,

$$g(y) = 2^{-\frac{\nu}{2}} \sum_{j=0}^{\infty} \frac{y^{(\nu/2)+j-1}\lambda^j}{\Gamma(\frac{\nu}{2}+j)2^{2j}j!}.$$

We next substitute $\nu = 2n+1$ and get

$$g(y) = 2^{-n-\frac{1}{2}} \sum_{j=0}^{\infty} \frac{y^{n+j-\frac{1}{2}}\lambda^j}{\Gamma(n+j+\frac{1}{2})2^{2j}j!}.$$

Next we use the identity (see [1, p.888])

$$\Gamma(n+\frac{1}{2}) = \frac{(2n)!\sqrt{\pi}}{2^{2n}n!},$$

and get

$$g(y) = (1/\sqrt{\pi})(2y)^{n-\frac{1}{2}} \sum_{j=0}^{\infty} \frac{(y\lambda)^j (n+j)!}{[2(n+j)]!j!}.$$

We then make the following change of variables $k = n+j$, and get

$$g(y) = (1/\sqrt{\pi})(2y)^{n-\frac{1}{2}} \sum_{k=n}^{\infty} \frac{(y\lambda)^{k-n}k!}{(2k)!(k-n)!}.$$

Note now that the Maclaurin series expansion of the hyperbolic cosine function is given by (see [1, p.41])

$$\cosh(y) = \sum_{j=0}^{\infty} \frac{y^{2j}}{(2j)!}.$$

Thus, it can be shown by induction that

$$\frac{\partial^n \cosh(\sqrt{\lambda y})}{\partial y^n} = \sum_{k=n}^{\infty} \frac{k!\lambda^k y^{k-n}}{(2k)!(k-n)!}.$$

Therefore,

$$g(y) = \frac{(2y/\lambda)^n}{\sqrt{2\pi y}} \frac{\partial^n \cosh(\sqrt{\lambda y})}{\partial y^n},$$

and this concludes the proof.

2) Remark: It can be seen from this theorem that for the special case $n = 0$ that corresponds to $\nu = 1$, we have

$$f(y) = \frac{\exp[-\frac{1}{2}(y+\lambda)]}{\sqrt{2\pi y}} \cosh(\sqrt{\lambda y})\cdot$$

Hence, throughout this paper, we will consider the case when $\nu > 1$ and odd.

B. Hyperbolic Theorem

For $\nu = 2n+1$, $n \in \mathbb{N}^*$, the non-central χ^2 distribution is given by

$$f(y) = \frac{\exp[-\frac{1}{2}(y+\lambda)]}{\sqrt{2\pi y}} \left(\frac{y}{\lambda}\right)^{\frac{n}{2}} \sum_{i=1}^{n} (-1)^{i-1} (\lambda y)^{(1-i)/2} h(n,i) \sinh_{n-i} \sqrt{\lambda y}, \quad (5)$$

where

$$\begin{aligned} \sinh_{n-i} \sqrt{\lambda y} &= \begin{cases} \sinh \sqrt{\lambda y} & \text{if } (n-i) \text{ is even;} \\ \cosh \sqrt{\lambda y} & \text{if } (n-i) \text{ is odd,} \end{cases} \\ &= \frac{1}{2}[\exp(\sqrt{\lambda y}) + (-1)^{n-i+1} \exp(-\sqrt{\lambda y})] \end{aligned}$$

and $h(n,i)$ is the two-dimensional function defined by the following recurrence relation

$$\begin{aligned} h(n+1, i+1) &= h(n,i)(n+i-1) + h(n, i+1); \\ h(n,n) &= h(n, n-1) = (2n-3)!!; \\ h(i,1) &= 1; \ i = 1, 2, \ldots, n; \\ h(n,i) &= 0; \ \text{for } i > n, \ i \le 0, \ n \le 0, \end{aligned}$$

where $n!!$ is the double factorial function defined as

$$n!! = \begin{cases} n(n-2)\ldots 3\cdot 1 & \text{if} \quad n > 0 \quad \text{is odd;} \\ n(n-2)\ldots 4\cdot 2 & \text{if} \quad n > 0 \quad \text{is even,} \\ 1 & \text{if} \quad n = -1, 0. \end{cases}$$

1) Proof: Having the result of Theorem II-A, it suffices to show that

$$\frac{\partial^n \cosh(\sqrt{\lambda y})}{\partial y^n} = \sum_{i=1}^{n} (-1)^{i-1} 2^{-n} \lambda^{\frac{1-i+n}{2}} y^{\frac{1-i-n}{2}} h(n,i) \sinh_{n-i} \sqrt{\lambda y}.$$

This can be proved by induction as follows. First, it is easy to show that the statement is true for $n = 1$. Next, assume that the statement is true for $n = k$ and prove that it is true for $n = k+1$. So we have

$$\begin{aligned} \frac{\partial^{k+1} \cosh(\sqrt{\lambda y})}{\partial y^{k+1}} &= \frac{\partial \partial^k \cosh(\sqrt{\lambda y})}{\partial y \partial y^k} \\ &= \frac{\partial}{\partial y} \left(\sum_{i=1}^{k} (-1)^{i-1} 2^{-k} \lambda^{\frac{1-i+k}{2}} y^{\frac{1-i-k}{2}} h(k,i) \sinh_{k-i} \sqrt{\lambda y} \right) \\ &= \sum_{i=1}^{k} \left((-1)^{i} 2^{-k-1} \lambda^{\frac{1-i+k}{2}} y^{\frac{-1-i-k}{2}} (k+i-1) h(k,i) \sinh_{k-i} \sqrt{\lambda y} \right) \\ &\quad + \sum_{i=1}^{k} \left((-1)^{i-1} 2^{-k-1} \lambda^{\frac{2-i+k}{2}} y^{\frac{-i-k}{2}} (k+i-1) h(k,i) \sinh_{k-i-1} \sqrt{\lambda y} \right) \end{aligned}$$

Next, apply the change of variables $l = i + 1$ in the first summation and get

$$\begin{aligned}
\frac{\partial^{k+1}\cosh(\sqrt{\lambda y})}{\partial y^{k+1}} &= \sum_{l=2}^{k+1}\left((-1)^{l-1}2^{-k-1}\lambda^{\frac{2-l+k}{2}}y^{\frac{-l-k}{2}}(k+l-2)h(k,l-1)\sinh_{k-l+1}\sqrt{\lambda y}\right)\\
&\quad+\sum_{i=1}^{k}\left((-1)^{i-1}2^{-k-1}\lambda^{\frac{2-i+k}{2}}y^{\frac{-i-k}{2}}(k+i-1)h(k,i)\sinh_{k-i-1}\sqrt{\lambda y}\right)\\
&= 2^{-k-1}\lambda^{\frac{1+k}{2}}y^{\frac{-1-k}{2}}\sinh_{k-2}\sqrt{\lambda y}\\
&\quad+\sum_{i=2}^{k}\left((-1)^{i-1}2^{-k-1}\lambda^{\frac{2-i+k}{2}}y^{\frac{-i-k}{2}}[(k+i-2)h(k,i-1)+h(k,i)]\sinh_{k-i+1}\sqrt{\lambda y}\right)\\
&\quad+(-1)^{k}2^{-k-1}\lambda^{\frac{1}{2}}y^{\frac{-l}{2}}(2k-1)h(k,k)\sinh_{0}\sqrt{\lambda y}\\
&= \sum_{i=1}^{k+1}(-1)^{i-1}2^{-k-1}\lambda^{\frac{2-i+k}{2}}y^{\frac{-i-k}{2}}h(k+1,i)\sinh_{k-i+1}\sqrt{\lambda y},
\end{aligned}$$

and this concludes the proof.

C. Recurrence Lemma

Let $h(n,i)$ be the function defined by the following recurrence relation

$$\begin{aligned}
h(n+1,i+1) &= h(n,i)(n+i-1)+h(n,i+1); &(6)\\
h(n,n) &= h(n,n-1) = (2n-3)!!;\\
h(i,1) &= 1;\ i = 1,2,\dots,n;\\
h(n,i) &= 0;\ \text{for } i>n,\ i\le 0,\ n\le 0,
\end{aligned}$$

then the solution is given by

$$h(n+1,i+1) = \frac{(n+i)!}{2^{i}i!(n-i)!} \quad n \ge i. \tag{7}$$

1) Proof: We start by plugging (7) in (6) and get

$$\begin{aligned}
h(n,i)(n+i-1)+h(n,i+1) &= \frac{(n+i-2)!(n+i-1)}{2^{i-1}(i-1)!(n-i)!}+\frac{(n+i-1)}{2^{i}i!(n-i-1)!}\\
&= \frac{(n+i-1)!(2i+n-i)}{2^{i}i!(n-i)!}\\
&= \frac{(n+i)!}{2^{i}i!(n-i)!}\\
&= h(n+1,i+1).
\end{aligned}$$

Next we check the constraints. So, from (7) we have

$$\begin{aligned}
h(n,n) &= \frac{(2n-2)!}{2^{n-1}(n-1)!}\\
&= (2n-3)!!,
\end{aligned}$$

where in the last step we applied the following identity

$$(2n-1)!! = \frac{(2n)!}{2^{n}n!}\cdot$$

We also have

$$\begin{aligned} h(n,n-1) &= \frac{(2n-3)!}{2^{n-2}(n-2)!} \\ &= [2(n-2)+1]!! \\ &= (2n-3)!!, \end{aligned}$$

where in the last step we applied the following identity

$$(2n+1)!! = \frac{(2n+1)!}{2^n n!}\cdot$$

Finally, we have

$$h(k,1) = \frac{(k-1)!}{(k-1)!} = 1,$$

and this concludes the proof.

D. Finite Sum Theorem

For $\nu = 2n+1$, $n \in \mathbb{N}^*$, the non-central χ^2 distribution is given by

$$f(y) = \frac{\exp[-\frac{1}{2}(y+\lambda)]}{\sqrt{2\pi y}} \left(\frac{y}{\lambda}\right)^{\frac{n}{2}} \sum_{i=0}^{n-1} (-1)^i (\lambda y)^{-\frac{i}{2}} \frac{(n+i)!(n-i)}{2^i i!(n-i)!(n+i)} \cosh_{n-i}\sqrt{\lambda y}, \tag{8}$$

where

$$\begin{aligned} \cosh_{n-i}\sqrt{\lambda y} &= \begin{cases} \cosh\sqrt{\lambda y} & \text{if } (n-i) \text{ is even;} \\ \sinh\sqrt{\lambda y} & \text{if } (n-i) \text{ is odd,} \end{cases} \\ &= \frac{1}{2}[\exp(\sqrt{\lambda y}) + (-1)^{n-i}\exp(-\sqrt{\lambda y})] \end{aligned}$$

1) Proof: The result of this theorem is reached by applying Lemma II-C to the result of Theorem II-B. Then some simple algebra including the change of variables $k = i - 1$ in the summation.

III. Summary

A finite sum representation of the probability density distribution (pdf) for a non-central χ^2 distribution in the case when the number of degrees of freedom ν is odd was presented. This finite sum has $(\nu+1)/2$ terms only instead of the infinite sum representation of the pdf for this distribution that is known in the literature. It would be of interest to extend this result to the case when ν is even.

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