On the Power Allocation of MIMO Channels

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Abstract— *In this paper we consider MIMO system with M_t transmitting and M_r receiving antennas, when channel state information (CSI) is known on the transmitter side. The Reyleigh fading channel propagation condition is assumed. In this case, the optimum transmission consists of allocating the transmitted power for each virtual sub-channel related to the corresponding eigenvalue of the channel matrix. The optimum power allocation is computed using the water pouring algorithm (WPA). However, a real-time implementation of the algorithm requires serious computational work, which is $O(M_t^2)$. We propose a modification to WPA that reduces the computational complexity to $O(M_t)$.*

Keywords: Array signal processing, computational complexity, MIMO system, water pouring algorithm

1. Introduction

With every day growing new wireless applications such as video streaming, multimedia, Internet, and so on, the demand for high-throughput wireless networks is ever increasing. Multiple-input-multiple-output (MIMO) technology that applies multiple antennas is an attractive way to increase data throughput, or channel capacity, of a wireless network [1], [2].

The main benefits of the multiple transmission and receiving is improving reception reliability by sending the same data from multiple antennas (spatial diversity) or increasing data rate by sending different data at the same time (spatial multiplexing) [3]. For the latter, MIMO system decomposes the propagation channel into orthogonal virtual spatial subchannels that transmit the data in parallel. The maximum overall channel capacity can be achieved when the full channel state information (CSI) is available at the transmitter. CSI can be acquired with explicit feedback for TDMA systems due to channel reciprocity or with radio frequency calibration for FDMA systems [2]. CSI helps allocate transmitted power among the transmitting antennas with an optimum power allocation principle referred to as *water pouring algorithm* (WPA) [4], [5]. According to this principle, the better the channel is, the more power it gets. As a result, the overall throughput of MIMO system becomes maximum.

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Despite the overall throughput of MIMO system is maximized, the computation of the channel power allocation using WPA is rather expensive due to high computational cost of the algorithm implementation, which is $O(M_t^2)$, where M_t is the number of transmitting antennas.

Furthermore, the up-to-date wireless networks imply that users can move with rather high speed causing highly non-stationary scenarios, such as the case of the military MONET. Moreover, CSI should also be updated in response to the changing scenario. When users movement is significant relatively to CSI updating rate, this can lead to serious MIMO performance degradation. To avoid this drawback, CSI updating rate must be fast enough to be able to track the changing scenario. Hence, WPA should continuously recompute the MIMO channels power allocation factors, by doing it in real time for the scenario changing tracking. This fact even greater intensifies the computational work that can dramatically decrease the benefits of MIMO systems. Therefore, reducing the computational complexity of WPA presents a real challenge. Thus, in this paper we propose an improvement of the water pouring algorithm by reducing the computational complexity from $O(M_t^2)$ to $O(M_t)$.

The rest of the paper is as follows. Section 2 introduces the MIMO channel model and discusses the corresponding power allocation using WPA, which is $O(M_t^2)$. Section 3 establishes a modified version of WPA that is $O(M_t)$. Section 4 presents a numerical example comparing both algorithms. Finally, concluding remarks are included in Section 5.

2. MIMO channel model and water pouring principle

Let us consider the MIMO system of M_t transmitting and M_r receiving antennas that operate in Reyleigh fading propagation channel (Fig 1.). The channel propagation coefficients h_{ij}, $i = 1, 2, \ldots, M_r$, $j = 1, 2, \ldots, M_t$ from any transmitting antenna to any receiving antenna are i.i.d. complex values $\mathcal{CN}(0, 1)$. Hence, the resulting channel matrix $\mathbf{H}_{\mathcal{N}} \in \mathbb{C}^{M_r \times M_t}$ is the collection of all channel propagation coefficients as follows

$$\mathbf{H}_{\mathcal{N}} = \begin{bmatrix} h_{11} & h_{12} & \ldots & h_{1M_t} \\ h_{21} & h_{22} & \ldots & h_{2M_t} \\ \vdots & \vdots & \ddots & \vdots \\ h_{M_r1} & h_{M_r2} & \ldots & h_{M_rM_t} \end{bmatrix}.$$

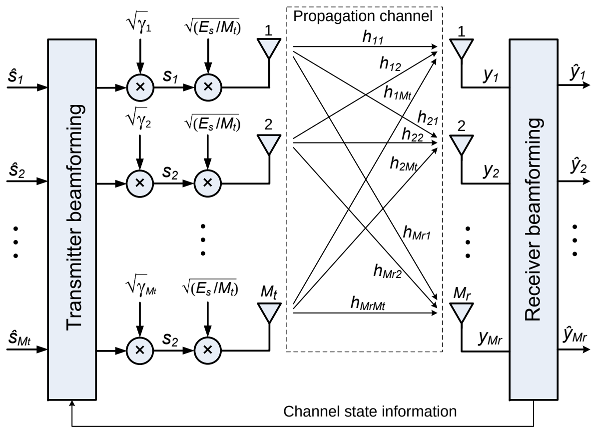

Fig. 1: MIMO system with adjustable channel power allocation factors

To approach more realistic propagation model, the elements of the matrix $\mathbf{H}_{\mathcal{N}}$ are expected to be correlated. The correlation is introduced by pre-multiplying the matrix $\mathbf{H}_{\mathcal{N}}$ by the square root of the receiver correlation matrix $\mathbf{R}_r$, and post-multiplying by the square root of the transmitter correlation matrix $\mathbf{R}_t$. In other words, we have

$$\mathbf{H} = \mathbf{R_r}^{1/2}\mathbf{H}_{\mathcal{N}}\mathbf{R}_t^{1/2}. \tag{1}$$

Assuming that $\mathbf{R}_r$ is the identity matrix, we consider the case with correlation at the transmitter side only, which depends on inter-element distances in the transmitting array. For the uniform linear array, the correlation coefficient r_{ij} between ith and jth transmitting antennas is modeled as

$$r_{ij} = J_0\left[2\pi(i-j)d/\lambda\right], \tag{2}$$

where $J_0(x)$ is the zero order Bessel function of the first kind, and d/λ is an inter-element distance to the carrier wavelength ratio.

Furthermore, we ignore the large scale propagation attenuation of the received signal, assuming that $\sum_{j=1}^{M_t} E\{|h_{ij}|^2\} = M_t, i = 1, 2, \ldots, M_r$, where $E\{\cdot\}$ is the expectation operator. This implies that each of the receiver antenna receives a power, which is equal to the total transmitted power E_s.

We assume that channel matrix is estimated at the receiver side, and the resulting channel state information (CSI) is retransmitted to the transmitter through a feedback channel. As a result, the singular value decomposition $\mathbf{H} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^\dagger$ could be available, where $\mathbf{U}$ and $\mathbf{V}$ are corresponding receiver and transmitter beamforming matrices with orthonormal properties, $\mathbf{\Sigma} = diag\{\sigma_1, \sigma_2, \ldots, \sigma_r\}$ is a diagonal matrix with the singular values entries, r is the rank of the matrix $\mathbf{H}$, and $\dagger$ is a conjugate and transpose symbol.

The transmitted signal $\mathbf{s}$ is a $M_t \times 1$ column vector with i.i.d. standard Gaussian entries $s_i,\ i = 1, 2, \ldots, M_t$. Therefore, the covariance matrix of the transmitted signal is

$$\begin{aligned}\mathbf{R}_{ss} &= E\left\{\mathbf{s}\mathbf{s}^\dagger\right\} \\ &= diag\left\{\gamma_1, \gamma_2, \ldots, \gamma_{M_t}\right\}\end{aligned} \tag{3}$$

where $\gamma_i = E\left\{|s_i|^2\right\}, i = 1, 2, \ldots, M_t$ is the ith subchannel power allocation factor, as shown in Fig. 1.

The received signal of MIMO system is given by

$$\mathbf{y} = \sqrt{\frac{E_s}{M_t}}\mathbf{H}\mathbf{s} + \mathbf{n} \tag{4}$$

where $\mathbf{y}$ is an $M_r \times 1$ column vector with the elements $y_i,\ i = 1, 2, \ldots, M_t$.

The corresponding covariance matrix of the received signal is

$$\mathbf{R_{yy}} = \frac{E_s}{M_t}\mathbf{H}\mathbf{R}_{ss}\mathbf{H}^\dagger + N_0\mathbf{I}_{M_r} \tag{5}$$

where N_0 is a white Gaussian noise power and $\mathbf{I}_{M_r}$ is the identity matrix of size M_r.

When the CSI is not available to the transmitter, more simple power allocation strategy is to set $\gamma_i = 1$ in (3), which yields the following resulting channel capacity

$$C = \log\det\left(\mathbf{I}_{M_r} + \beta\mathbf{\Lambda}\right), \tag{6}$$

where $\beta = \frac{E_s/N_0}{M_t}$ is the received signal-to-noise-ratio per each receiving antenna normalized by the number of transmitting antennas, and $\mathbf{\Lambda} = diag\left\{\lambda_1, \lambda_2, \ldots, \lambda_{M_r}\right\}, \lambda_i \geq \lambda_{i+1}$ is generated through the eigenvalue decomposition of the semidefine matrix $\mathbf{H}\mathbf{H}^\dagger = \mathbf{Q}\mathbf{\Lambda}\mathbf{Q}^\dagger$ [6]; thus we have $\sqrt{\lambda_i} = \sigma_i$ for $i = 1, 2, \ldots, r$.

Unfortunately, such kind of power allocation strategy does not allow to achieve the maximum channel throughput. Optimization of the MIMO channel capacity requires the solution of the following maximization problem subject to transmitter power constraint [7]

$$C = \max_{Tr(\mathbf{R}_{ss})=M_t} \log\det\left(\mathbf{I}_{M_r} + \beta\mathbf{H}\mathbf{R}_{ss}\mathbf{H}^\dagger\right), \tag{7}$$

where $Tr(\cdot)$ means the matrix trace operator.

The solution of (7) leads to the water-pouring principle that maximizes the MIMO channel capacity by allocating more power to the channel that is in good condition and less or none at all to the bad ones. By doing so, the resulting MIMO channel capacity is maximized and (7) can be expressed as the sum of the individual parallel SISO (single-input-single-output) channel capacities

$$C = \max_{\sum_{i=1}^r \gamma_i = M_t} \sum_{i=1}^{r} \log\left(1 + \beta\gamma_i\lambda_i\right), \tag{8}$$

where γ_i is the ith subchannel transmit power allocation factor that is found by WPA, such that $\sum_{i=1}^r \gamma_i = M_t$.

The optimum energy allocation procedure using WPA was introduced in [4]. To start the WPA we assume that CSI in the transmitter side is received, the matrix $\mathbf{\Lambda}$ is available, and the rank r of the matrix $\mathbf{H}$ is known.

The algorithm consists of the following steps [4], [5].

1) Initialization. Given the values M_t, M_r, β, r, and $\mathbf{\Lambda}$.
2) For $p = 1, 2, \ldots, r$ calculate the constant
$\mu_p = \frac{1}{r-p+1}\left(M_t + \beta^{-1}\sum_{i=1}^{r-p+1}\lambda_i^{-1}\right)$.
3) For $j = 1, 2, \ldots, r-p+1$ allocate the power for j subchannel as
$\gamma_j = \mu_p - \beta^{-1}\lambda_j^{-1}$.
end j
4) If the $\gamma_{r-p+1} < 0$, then put $\gamma_{r-p+1} = 0$ and return to the Step 2 assigning $p = p+1$, otherwise the algorithm ends. The resulting values of γ_j correspond to the optimum power allocation strategy.

According to the above presented WPA, the optimum subchannel power (likewise the energy) allocation requires recurrent computation, and the computational work in general case is $O(M_t^2)$. Despite the optimum energy allocation for all subchannels that maximize the channel capacity, the real-time implementation of WPA is rather restricted due to hight computational complexity. Therefore, for practical reasons, WPA is often replaced by the simplest solution, where the transmitted energy is distributed uniformly among all transmitting antennas [8]. Since this practice avoids using WPA, it reduces the channel capacity.

3. Water pouring algorithm modification

Despite the matrix $\mathbf{HH}^\dagger$ is semidefine, i.e. $\lambda_i \geq 0, i = 1, 2, \ldots, M_t$, negative values of γ_{r-p+1} in step 3 of WPA can appear frequently, and as a consequence, some channels should be discarded. As a matter of fact, the larger the condition number of $\mathbf{HH}^\dagger$, the more channels are discarded, where the condition number is define as λ_1/λ_r, where $\lambda_r \neq 0$, since r is the rank of the matrix.

For example, at low signal-to-noise-ratio the condition number is high enough, and the solution is reduced to the case of beamforming, i.e., only the channel with the highest eigenvalue is in use. Thus, the traditional algorithm keeps repeating itself by increasing p by one, and computes the new values of μ_p and γ_j for every value of p. Since the convergence of WPA for such case is slow, this situation is unfavorable in computational sense. On the other hand, at high signal-to-noise-ratio for all subchannels yields small condition number of $\mathbf{HH}^\dagger$ and all eigenvalues are close to each other. This leads to almost uniform power allocation, which is more favorable from the point of view of computational work, since the convergence is fast.

Since $\mathbf{H}$ is random, the corresponding condition number, which affects the performance of WPA, is also random, and as a consequence unpredictable. Thus, from WPA implementation point of view, the upper bound computational work of $O(M_t^2)$ has to be taken into consideration. Furthermore, in the highly non-stationary communication scenario, $\mathbf{H}$ changes frequently, and any computational delay (which is proportional to the computational work) can cause the information about energy allocation to be obsolete. This in turn, degrades the channel capacity. Therefore, there is the need for minimizing the computational work as much as possible.

To this end, we propose the modified WPA that requires only $O(M_t)$ operations in the worst case, and is independent of the propagation channel condition number. Accordingly, the term $\gamma_j = \mu_p - \beta^{-1}\lambda_j^{-1}$ presented in Step 3 of WPA, is negative when

$$\mu_p < \beta^{-1}\lambda_j^{-1} \tag{9}$$

which gives us a clue to jump directly to the value of p that provides non-negative solution of WPA.

To develop the modified WPA we introduce the matrix $\mathbf{\Psi} = diag\{\psi_1, \psi_2, \ldots, \psi_r\}$ with the reciprocal elements of the $\lambda_i, i = 1, 2, \ldots, r$, i.e. $\mathbf{\Psi} = \mathbf{\Lambda^{-1}}$. We rewrite (9) for the case when $j = r-p+1$ as

$$\begin{aligned}&\frac{1}{r-p+1}\left[M_t + \beta^{-1}\left(\psi_1 + \psi_2 + \ldots + \psi_{r-p+1}\right)\right]\\ &< \beta^{-1}\psi_{r-p+1}.\end{aligned} \tag{10}$$

Multiplying both sides of (10) by $\beta(r-p+1)$, and then removing the term ψ_{r-p+1} from the left side to the right one we get

$$\beta M_t + \psi_1 + \psi_1 + \ldots + \psi_{r-p} < (r-p)\psi_{r-p+1}. \tag{11}$$

Since $\beta = \frac{E_s/N_0}{M_t}$, we can rewrite the decision rule as

$$\alpha_p \lesseqgtr 0, \tag{12}$$

where

$$\alpha_p = E_s/N_0 + \sum_{i=1}^{r-p}\left(\psi_i - \psi_{r-p+1}\right). \tag{13}$$

Hence, negative α_p in (12) indicates that the power allocation algorithm for a given p has a negative solution at least for a channel $r-p+1$, and therefore it does not make sense to compute γ_j for $j = 1, 2, \ldots, r-p+1$. On the other hand, the positive value α_p indicates that WPA for channel $r-p+1$ has a positive solution, i.e. $\gamma_{r-p+1} > 0$, as well as for the previous channels, i.e. $\gamma_j > 0$, where $j < r-p$; for the remaining channels the power allocation algorithm has a negative solution, i.e. $\gamma_j < 0$, where $j > r-p+1$, and therefore, those channels should be discarded. The case $\alpha_p = 0$ is a border line decision case that can be joined to one of the aforementioned cases.

All in all, we can present the following modified water pouring algorithm (MWPA)

1) Given the values M_t, M_r, β, r and $\mathbf{\Psi}$.
2) For $p = 1, 2, \ldots, r$ calculate α_p with (13)
If $\alpha_p \leq 0$, then $p = p+1$,
If $\alpha_p > 0$, then $p_{opt} = p$. Compute
$\mu_{p_{opt}} = \frac{1}{r-p_{opt}+1}\left(M_t + \beta^{-1}\sum_{i=1}^{r-p_{opt}+1}\psi_i\right)$,
and go to Step 3
end p

3) For $j = 1, 2, \ldots, r - p_{opt} + 1$ allocate the power for j subchannel as
   $\gamma_j = \mu_{p_{opt}} - \beta^{-1}\psi_i$.
   end j
4) Discard the channels $j > r - p_{opt} + 1$ by allocating zero power on each. Resulting values of γ_j correspond to the optimum power allocation strategy.

According to this modified algorithm, the optimum subchannel power allocation *does not require* recurrent computation. The upper bound of the computational work can be found when we set $r = M_t$ and $p_{opt} = 1$. For this case WPA requires M_t multiplications and M_t summations.

4. Numerical example

The objective of this simulation study is to compare numerically the performance of the conventional WPA and its modification. Accordingly, we consider uniform linear arrays with $M_t = M_r = 4$, the ratio $E_s/N_0 = 2$, and $d/\lambda = 1/2$. The first row of the transmitting matrix $\mathbf{R}_t$ according to (2) is $\mathbf{r}_1 = [1.0000; -0.3042; 0.2203; -0.1812]$. The corresponding real and image parts of the channel matrix $\mathbf{H}$ as well as its eigenvalues matrix $\mathbf{\Lambda}$ presented below.

$$Re\{\mathbf{H}\} = \begin{bmatrix} 0.8459 & 0.8411 & 0.0196 & 0.2112 \\ -1.3603 & -1.9453 & -0.7849 & -0.1938 \\ 0.4768 & 0.2510 & -0.1228 & -0.2380 \\ -0.9921 & -0.7571 & 0.1009 & -0.1963 \end{bmatrix},$$

$$Im\{\mathbf{H}\} = \begin{bmatrix} -0.0484 & -0.8005 & -0.0541 & 0.0928 \\ 1.7752 & 1.1112 & 1.9234 & 1.9799 \\ 1.5349 & 1.4888 & 1.5180 & 2.0033 \\ -1.1970 & -1.5465 & -1.0858 & -1.5249 \end{bmatrix},$$

$$\mathbf{\Lambda} = diag\{36.4641 \quad 3.6861 \quad 0.3671 \quad 0.0870\}.$$

Since all eigenvalues are positive, the matrix $\mathbf{H}$ has full rank, i.e. $r = M_t = 4$. The channel power allocation with the conventional WPA gives the values μ_p and γ_j, $j = 1, 2, 3, 4$, presented in Table 1; the power allocation diagram depicted in Fig 2, where $n = 1, 2, 3, 4$ is a virtual channel number, and shadow areas related to the positive values of γ_n .

Table 1: Numerical example of WPA solution

p	μ	γ_1	γ_2	γ_3	γ_4
$p=1$	8.2595	8.2047	7.7170	2.8108	-14.7325
$p=2$	3.3487	3.2939	2.8061	-2.1000	0
$p=3$	2.2987	2.2439	1.7561	0	0

Table 1 as well as Fig 2 shows, WPA requires three iterations with calculating nine values of γ_j, but only two values of γ_j is required, when $p = 3$ (last row in the Table 1). All other are unuseful and they computations are redundant.

In the same time, the modified algorithm give us the following result. The values α_p is -29.4649, -3.1500, and 1.7561 for p = 1, 2, and 3, respectively. Therefore $\alpha_{opt} =$ 3. Accordingly, we get γ_j is 2.2439, 1.7561, 0.0000, and 0.0000 for j = 1, 2, 3, and 4, respectively.

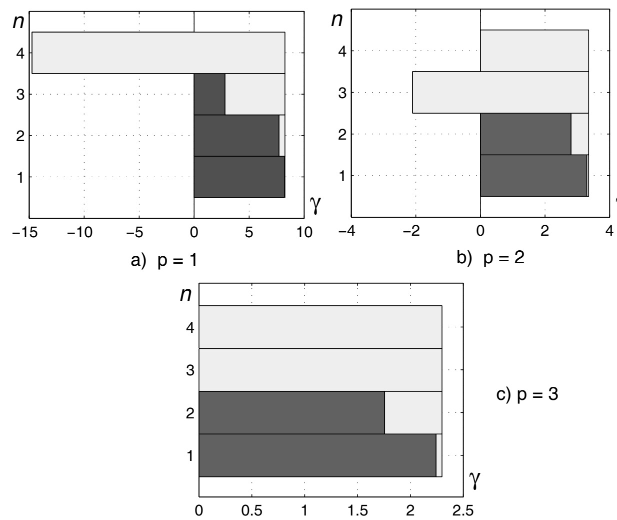

Fig. 2: Power allocation diagram

It requires to calculate three times α_p (easy, because according to (13) contains only summation operations), then to do three comparisons (simple logical OR operations), and afterward to calculate only two significant values, γ_1 and γ_2, versus nine values in the counterpart. Hence, the gain of MWPA over conventional version in the presented numerical example is about four times.

5. Conclusion

We proposed an improvement to the well-known water pouring algorithm that helps to allocate optimum transmitted power among the transmitting antennas of a MIMO system. The improvement increases the efficiency of the algorithm from $O(M_t^2)$ to $O(M_t)$, where M_t is the number of transmitting antennas. This in turn reduces the cost of implementation of multi-antenna MIMO systems.

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